Completions of Dieudonné complexes and Dieudonné algebras

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1 Introduction

Where we're going: given an $\mathbb{F}_p$-algebra R, we want to construct a canonical complex computing its crystalline cohomology. Our first attempt is to lift R to characteristic zero, along with a Frobenius map, and take the p-completed de Rham complex of this lift $\widetilde{R}$. This computes the right cohomology, but it has an obvious flaw: as a complex, it depends on our choice of lift.

To fix this problem, we will view $\widehat{\Omega}_{\widetilde{R}}$ as a Dieudonné complex (or better, a Dieudonné algebra—you can probably imagine roughly what this means, and we'll define it later today), and we will apply two homological algebra operations to this: saturation and ("V-adic") completion (= strictification). These operations will enlarge our complex considerably. We will eventually show that the output of this process satisfies a universal property depending only on R, which implies that it is well-defined and functorial in R.

The main goals for this talk are as follows: introduce the last completion operation $\mathcal{W}$ and check that it behaves as expected, define Dieudonné algebras, and finally discuss some properties of the de Rham complex and its p-adic completion as Dieudonné algebras.

2 The completion of a Dieudonné complex

Let M^* be a saturated Dieudonné complex. Last time we saw how to endow M^* with a Verschiebung operator V, a homomorphism of graded abelian groups satisfying $VF = FV = p$, $Vd = pdV$, and $FdV = d$. This allows us to make the following definition:

Definition 2.1. For M^* a saturated Dieudonné complex, we set $\mathcal{W}_r(M)^* = M^*/(\operatorname{im} V^r + \operatorname{im} dV^r)$ for $r \geq 0$. These come equipped with quotient maps $\mathbb{R} : \mathcal{W}_{r+1}(M) \to \mathcal{W}_r$, which allow us to form the *completion*

$$\mathcal{W}(M)^* = \varprojlim(\cdots \to \mathcal{W}_2(M)^* \to \mathcal{W}_1(M)^* \to \mathcal{W}_0(M)^* = 0). \quad (1)$$

*Notes for a talk in Berkeley's number theory seminar, on Bhatt-Lurie-Mathew's paper *Revisiting the de Rham-Witt complex.*

Clearly, this is a complex of abelian groups equipped with a morphism $\rho_M : M^* \to \mathcal{W}(M)^*$. We have several claims:

1. $\mathcal{W}(M)^*$ is naturally equipped with the structure of a Dieudonné complex, and is functorial in M^*.

2. $\mathcal{W}(M)^*$ is saturated.

3. Define a saturated Dieudonné complex M^* to be *strict* if $M^* \to \mathcal{W}(M)^*$ is an isomorphism. Then $\mathcal{W}(M)^*$ is strict, and it is universal among strict Dieudonné complexes equipped with a map from M^*.

4. The category $\mathbf{DC}_{\text{str}}$ of strict Dieudonné complexes is equivalent to the category of so-called *strict Dieudonné towers* (to be defined soon), which will package together the data of $\mathcal{W}_r(M)^*$ for all r without assuming that they actually come from some M^*.

5. There is a "generalized Cartier isomorphism" relating $\mathcal{W}_r(M)^*$ to $H^*(M^*/p^r M^*)$.

Before checking these properties, let's look at two examples.

First, let $A = A[0]$ be any abelian group viewed as a complex concentrated in degree zero. Then any group map $F : A \to A$ gives A the structure of a Dieudonné complex. This is saturated if and only if A is p-torsion-free and F is an isomorphism. If so, then $\operatorname{im} V^r + \operatorname{im} dV^r = p^r A$, so $\mathcal{W}(A)^*$ is the p-adic completion.

A more complicated example is the *free strict Dieudonné complex on one generator x*. We set

$$M_0 = \{\sum_{m \geq 0} a_m F^m x + \sum_{n>0} b_n V^n x : a_m, b_n \in \mathbb{Z}_p, a_m \to 0\} \tag{2}$$

$$M_1 = \{\sum_{m \geq 0} c_m F^m dx + \sum_{n>0} d_n dV^n x : c_m, d_n \in \mathbb{Z}_p, c_m \to 0\} \tag{3}$$

and

$$d\left(\sum_{m \geq 0} a_m F^m x + \sum_{n>0} b_n V^n x\right) = \sum_{m \geq 0} p^m a_m F^m dx + \sum_{n>0} b_n dV^n x. \tag{4}$$

(Note that there is no condition on b_n or d_n, because V^n and dV^n already converge to 0.) Then $M^* = (M_0 \xrightarrow{d} M_1)$ is a strict Dieudonné complex, and it is freely generated by $x = F^0 x \in M^0$ in the sense that for any $N^* \in \mathbf{DC}_{\text{str}}$, evaluation on x gives a bijection

$$\operatorname{Hom}_{\mathbf{DC}}(M^*, N^*) \to N^0. \tag{5}$$

Let's start checking our claimed properties. First, in order for $\mathcal{W}(M)^*$ to be a Dieudonné complex, we must endow it with a Frobenius. To do so, we note that since $FV = p$ and $FdV = d$, F maps $\operatorname{im} V^r + \operatorname{im} dV^r$ into $\operatorname{im} V^{r-1} + \operatorname{im} dV^{r-1}$ and thus induces a unique map $\mathcal{W}_r(M)^* \to \mathcal{W}_{r-1}(M)^*$. Passing to the inverse limit gives a Frobenius map $F : \mathcal{W}(M)^* \to \mathcal{W}(M)^*$, which satisfies the relation $dF = pFd$ because this is true at all finite levels. This makes $\mathcal{W}(M)^*$ a Dieudonné complex.

For future reference, we note the V-analogue of this as well: V maps $\operatorname{im} V^n + \operatorname{im} dV^n$ into $\operatorname{im} V^{n+1} + \operatorname{im} dV^{n+1}$ (because $Vd = pdV$), so we get a map $V : \mathcal{W}_r(M)^* \to \mathcal{W}_{r+1}(M)^*$, which likewise passes to the limit.

Functoriality is clear: given $f : M^* \to N^*$, we get maps $f : \mathcal{W}_r(M^*) \to \mathcal{W}_r(N^*)$ for all r, compatible with the respective d and F maps. Then f passes to the limit, as does its compatibility with d and F.

2.1 Strict Dieudonné towers

At this point, it is useful to introduce the notion of a strict Dieudonné tower. This will abstract away the properties of the system $(\mathcal{W}_r(M)^*)$, and it will often be useful to work with the tower rather than the completion $\mathcal{W}(M)^*$.

Definition 2.2. A *strict Dieudonné tower* is an inverse system of complexes of abelian groups

$$\cdots \to X_2^* \xrightarrow{R} X_1^* \xrightarrow{R} X_0^*, \tag{6}$$

equipped with maps of graded groups $F : X_{r+1}^* \to X_r^*$ and $V : X_r^* \to X_{r+1}^*$, satisfying the following properties:

1. X_0^* is the zero complex.
2. $R : X_{r+1}^n \to X_r^n$ is surjective for all r, n.
3. $dF = pFd$.
4. F, V, and R all commute with each other.
5. $FV = VF = p \cdot \mathrm{id}$.
6. For all $x \in X_r^*$ with $dx \in pX_r^*$, $x \in \operatorname{im}(F : X_{r+1}^* \to X_r^*)$.
7. The kernel of $R : X_{r+1}^* \to X_r^*$ equals the p-torsion subgroup of X_{r+1}^*.
8. The kernel of $R : X_{r+1}^* \to X_r^*$ equals $\operatorname{im}(V^r) + \operatorname{im}(dV^r)$, where V^r and dV^r are the maps $X_1^* \to X_{r+1}^*$.

A morphism of strict Dieudonné towers is a family of morphisms $X_r^n \to Y_r^n$ compatible with d, R, F, and V. We write **TD** for the resulting category.

Proposition 2.3. *(2.6.2) If M^* is a saturated Dieudonné complex, then $(\mathcal{W}_r M)^*$ is a strict Dieudonné tower.*

The only nontrivial properties are (6) and (7). To prove these we use an easy lemma:

Lemma 2.4. *(2.6.3) If M^* is saturated and $x \in M^*$ satisfies $d(V^r x) \in pM^{*+1}$ for some r, then $x \in \operatorname{im} F$.*

Proof. Since $d(V^r x) \in pM^{*+1}$, we have $dx = F^r d(V^r x) \in pM^{*+1}$, and then $x \in \operatorname{im} F$ because M^* is saturated. □

Proof of proposition. Property 7 amounts to saying that for $x \in M^*$, we have $x \in \operatorname{im} V^r + \operatorname{im} dV^r$ if and only if $px \in \operatorname{im} V^{r+1} + \operatorname{im} dV^{r+1}$. The forward implication is clear:

$$p(V^r a + dV^r b) = V^r(pa) + dV^r(pb) \tag{7}$$

$$= V^{r+1}(Fa) + dV^{r+1}(Fb). \tag{8}$$

For the reverse implication[1], suppose $px = V^r a + dV^r b$ for some a, b. Taking d of both sides, we see that $d(V^r a)$ is a multiple of p. By the lemma, it follows that $a = F\widetilde{a}$ for some $\widetilde{a}$. But then $px = pV^{r-1}\widetilde{a} + dV^r b$, so $dV^r b$ is also divisible by p, and thus $b = F\widetilde{b}$ for some $\widetilde{b}$. We conclude that $px = pV^{r-1}\widetilde{a} + pdV^{r-1}\widetilde{b}$. Since M^* is p-torsion-free, we are done.

For property 6, suppose $\overline{x} \in \mathcal{W}_r(M)^*$ has the property that $dx \in p\mathcal{W}_r(M)^*$. Choosing a representative $x \in M^*$, this means that $dx = py + V^r a + dV^r b$ for some $a, b, y \in M^*$. Taking d of both sides, we see as before that $d(V^r a)$ is divisible by p and so $a = F\widetilde{a}$ for some $\widetilde{a}$. Then we have

$$d(x - V^r b) = p(y + V^{r-1}\widetilde{a}), \tag{9}$$

so the fact that M^* is saturated implies that $x - V^r b = F(z)$ for some $z \in M^*$. But this reduces to $\overline{x} \in \mathcal{W}_r(M)^*$, so $\overline{z} \in \mathcal{W}_{r+1}(M)^*$ maps to $\overline{x}$, and we are done. □

Proposition 2.5. *Let (X_r^*) be a strict Dieudonné tower, and let $X^* = \lim_{\leftarrow r} X_r^*$ be equipped with its natural Frobenius. Then (X^*, F) is a saturated Dieudonné complex.*

Proof. First of all, it is a Dieudonné complex, because the identity $dF = pFd$ passes to the limit. To see that it is p-torsion-free, take any $(x_r)_r$ in the inverse limit with $p(x_r) = 0$. By property 7, we have $x_{r-1} = R(x_r) = 0$ for all r.

It now remains to prove the main property of saturated complexes: that F gives a bijection from X^* to $\{x \in X^* : dx \in pX^{*+1}\}$. Note that F is injective, because $VF = p$ is. For surjectivity, suppose $x = (x_i), x_i \in X_i^*$ has $dx \in pX^*$. By property 6, each x_i equals $F(y_i)$ for some $y_i \in X_i^*$. These y_i are *not* compatible in general—but they are compatible after applying F, thus after applying $VF = p$, thus after applying $R : X_i^* \to X_{i-1}^*$ by property 7. So the inverse system $(R(y_{i+1}) \in X_i^*)$ is compatible and has the desired image under F. □

2.2 Homological properties of $\mathcal{W}_r(M)^*$

Our next goal is to show that the completion map $M^* \to \mathcal{W}(M)^*$ is a *strictification*; that is, a universal morphism to a strict Dieudonné complex. In the process, we will prove some homological properties of $\mathcal{W}_r(M)^*$ which will be useful later, and which will shed some light on the importance of taking the V-adic completion.

[1] This proof is surprisingly reminiscent of the standard proof that $\sqrt{p}$ is irrational!

Proposition 2.6. *(Generalized Cartier isomorphism) Let M^* be a saturated Dieudonné complex and $r \geq 0$. The map $F^r : M^* \to M^*$ induces an isomorphism of graded groups*

$$\mathcal{W}_r(M)^* \to H^*(M^*/p^r M^*). \tag{10}$$

Proof. Given an element $\overline{x} \in \mathcal{W}_r(M)^*$ represented by $x \in M^*$, we map $\overline{x}$ to the cohomology class $[F^r x] \in H^*(M^*/p^r M^*)$; note that this is a cycle because $dF^r x = p^r F^r dx \in p^r M^{*+1}$. This is well-defined because $[V^r x] \mapsto [F^r V^r x] = [p^r x] = 0$ and $[dV^r x] \mapsto [F^r dV^r x] = [dx] = 0$. To prove injectivity, note that $[F^r x] = 0$ implies that $F^r x = p^r y + dz$, so $F^r(x - V^r y - dV^r z) = 0$. But F^r is injective, so x must vanish in $\mathcal{W}_r(M)^*$. Surjectivity amounts to the statement that $F^r : M^* \to \{x \in M^* : dx \in p^r M^*\}$ is surjective. In fact it is a bijection; this is the definition of saturation in the case $r = 1$, and follows from induction in general. □

Warning: this isomorphism does *not* pass to the inverse limit. In fact it is not even compatible with the obvious maps $R : \mathcal{W}_r(M)^* \to \mathcal{W}_{r-1}(M)^*$ and $H^*(M^*/p^r M^*) \to H^*(M^*/p^{r-1}M^*)$, since the map at rth level is induced by F^r. It is compatible if one uses the map $F : \mathcal{W}_r(M)^* \to \mathcal{W}_{r-1}(M)^*$, but of course this gives a different inverse system.

This proposition tells us that $\mathcal{W}_r(M)^*$ has a remarkable property: the complex $\mathcal{W}_r(M)^*$ up to isomorphism "encodes" the complex $M^*/p^r M^*$ up to quasi-isomorphism. This is extremely useful for our purposes: when constructing the de Rham-Witt complex of a ring R, we begin with the de Rham complex of a lift $\widetilde{R}$, which is "the right complex up to quasi-isomorphism", and applying the strictification will help us isolate "the right complex on the nose". One way to make this precise is with the following rigidity result:

Corollary 2.7. *Let $f : M^* \to N^*$ be a morphism in $\mathbf{DC}_{\mathrm{sat}}$. The following conditions are equivalent:*

1. *f induces a quasi-isomorphism $M^*/pM^* \to N^*/pN^*$.*
2. *f induces an isomorphism $\mathcal{W}_1(M)^* \xrightarrow{\sim} \mathcal{W}_1(N)^*$.*
3. *For all $r \geq 0$, f induces a quasi-isomorphism $M^*/p^r M^* \to N^*/p^r N^*$.*
4. *For all $r \geq 0$, f induces an isomorphism $\mathcal{W}_r(M)^* \xrightarrow{\sim} \mathcal{W}_r(N)^*$.*

Proof. The proposition implies (1) $\iff$ (2) and (3) $\iff$ (4). The implication (3) $\implies$ (1) is trivial, and (1) $\implies$ (3) can be proved by induction on r. □

Proposition 2.8. *(2.7.6): Let M^* be a saturated Dieudonné complex. Then $\mathcal{W}(M)^*$ is strict; i.e. the map $\rho_{\mathcal{W}(M)^*} : \mathcal{W}(M)^* \to \mathcal{W}(\mathcal{W}(M))^*$ is an isomorphism.*

Proof sketch. One can check directly that $\rho_{\mathcal{W}(M)} = \mathcal{W}(\rho_M)$. So it suffices to check that $\mathcal{W}_r(\rho_M) : \mathcal{W}_r(M)^* \to \mathcal{W}_r(\mathcal{W}(M))^*$ is an isomorphism for each r. By the previous corollary, it suffices to check this for $r = 1$. This can be done by hand. □

One can then prove that $\rho_{M^*} : M^* \to \mathcal{W}(M)^*$ is in fact the *initial* map from M^* to a strict Dieudonné complex, and so the completion functor $\mathbf{DC}_{\mathrm{sat}} \to \mathbf{DC}_{\mathrm{str}}$ is left-adjoint to the forgetful functor. We omit the proof.

At this point, one can show that $(\mathcal{W}_r(-)) : \mathbf{DC}_{\rm str} \to \mathbf{TD}$ is an equivalence of categories: the reverse equivalence is $\lim_{\leftarrow}$, and the only thing to check is that a strict Dieudonné tower $(X_r)^*$ is naturally isomorphic to $(\mathcal{W}_r(\lim_{\leftarrow i} X_i^*))$.

The following results continue our theme of comparing M^* with $\mathcal{W}(M)^*$.

Proposition 2.9. *For $M^* \in \mathbf{DC}_{\rm sat}$, the map $\rho_M : M^* \to \mathcal{W}(M)^*$ induces quasi-isomorphisms $M^*/p^r M^* \to \mathcal{W}(M)^*/p^r\mathcal{W}(M)^*$ for every r.*

Proof sketch. We stated this with "$/p^r$" replaced with "$\mathcal{W}_r$" in the proof of the last proposition. Apply the equivalence (3) $\iff$ (4) in the previous corollary to that. □

The following result extends the last in two ways: it deals directly with the inverse limit (using a p-completion hypothesis), and applies to the strictification $M^* \to \mathcal{W}\,\mathrm{Sat}(M^*)$ of a not necessarily saturated Dieudonné complex M^*.

Corollary 2.10. *Let M^* be a Dieudonné complex of Cartier type (not necessarily saturated), and suppose each M^n is p-adically complete. Then the strictification map $M^* \to \mathcal{W}\,\mathrm{Sat}(M^*)$ is a quasi-isomorphism.*

Proof. Both the domain and codomain are complexes of p-complete, p-torsion-free abelian groups,[2] so it suffices[3] to show that $M^*/pM^* \to \mathcal{W}\,\mathrm{Sat}(M^*)/p\mathcal{W}\,\mathrm{Sat}(M^*)$ is a quasi-isomorphism. But this factors as

$$M^*/pM^* \to \mathrm{Sat}(M^*)/p\,\mathrm{Sat}(M^*) \to \mathcal{W}\,\mathrm{Sat}(M^*)/p\mathcal{W}\,\mathrm{Sat}(M^*), \qquad (11)$$

where the first map is a quasi-isomorphism by the Cartier criterion (the main theorem proved in Dan's talk last week), and the second is the $r = 1$ case of the previous proposition. □

3 Dieudonné algebras

Definition 3.1. A *commutative differential graded algebra (cdga)* is a complex (A^*, d) with the structure of a graded ring such that:

a) Multiplication is graded-commutative: for $x \in A^m$ and $y \in A^n$, $xy = (-1)^{mn} yx \in A^{m+n}$.

b) If $x \in A^n$ with n odd, then $x^2 = 0 \in A^{2n}$. (This is automatic if A^* has no 2-torsion.)

c) The Leibniz rule $d(xy) = (dx) \cdot y + (-1)^m x \cdot dy$ for $x \in A^m$.

Definition 3.2. A *Dieudonné algebra* (A^*, d, F) is a cdga (A^*, d) equipped with a graded ring homomorphism $F : A^* \to A^*$ such that:

1. $dF = pFd : A^* \to A^{*+1}$.

[2] TO DO: clarify the subtlety in saying that $\mathcal{W}\,\mathrm{Sat}(M)^n$ is p-complete.
[3] TO DO: check this!

2. $A^n = 0$ when $n < 0$.

3. For $x \in A^0$, $Fx \equiv x^p \pmod{p}$.

These form a category **DA**, where the morphisms are morphisms of graded rings compatible with d and F. Note that we have an obvious forgetful functor $\mathbf{DA} \to \mathbf{DC}$.

Reinterpretation: We can endow the category **DC** with a tensor product, given by the usual tensor product of complexes (with $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$ for x homogeneous) with $F(x \otimes y) := F(x) \otimes F(y)$. This makes **DC** into a symmetric monoidal category, where the symmetry is given by the isomorphism

$$A^* \otimes B^* \xrightarrow{\sim} B^* \otimes A^* \tag{12}$$

$$a \otimes b \mapsto (-1)^{|a||b|} b \otimes a. \tag{13}$$

Then a Dieudonné algebra is a commutative algebra object in the category **DC**, where the "commutativity" diagram encodes graded commutativity, and **DA** is the full subcategory of CommAlg(**DC**) where conditions 2, 3, and b above hold. (Note that strict Dieudonné complexes are automatically torsion-free, by fiat for p-torsion and by p-completeness for ℓ-torsion, so condition b comes for free when A^* is strict.)

Remark: For A^* a Dieudonné algebra, $\eta_p(A)^*$ is a Dieudonné subalgebra of A^*. The only subtle part is that $Fx \equiv x^p \pmod{p(\eta_p A^0)}$, because $\eta_p A^0 \neq A^0$ in general. To show that it works, suppose $x \in (\eta_p A)^0$, so that $dx = py$ for $p \in A^1$. Then since A^* is a Dieudonné algebra, we have $Fx = x^p + pz$, for some $z \in A^0$. To show that $z \in (\eta_p A)^0$ (i.e. $dz \in pA^1$), we calculate

$$p(dz) = d(pz) = d(Fx - x^p) = pF(dx) - px^{p-1}dx = p^2 Fy - p^2 x^{p-1} y \tag{14}$$

and cancel p's.

Moreover, last week we saw that Frobenius operators on a p-torsion-free complex A^* correspond to maps $\alpha : A^* \to \eta_p A^*$ of complexes, where $\alpha = p^n F$ in degree n. The same discussion goes through in this complex, where the maps α are required to be cdga homomorphisms:

$$\{\text{graded ring homomorphisms } F : A^* \to A^* \text{ with } dF = pFd\}$$
$$\updownarrow$$
$$\{\text{cdga homomorphisms } \alpha : A^* \to \eta_p A^*\}$$

Example: Let R be an $\mathbb{F}_p$-algebra, and $W(R) = W(R)[0]$ its ring of Witt vectors, as a cdga concentrated in degree 0. Then the Witt vector Frobenius $F : W(R) \to W(R)$ makes $W(R)$ a Dieudonné algebra. This is saturated as a Dieudonné complex if and only if F is a bijection, if and only if R is perfect. In this case, $W(R)$ is also p-complete and $p^n W(R) = \operatorname{im} V^n + \operatorname{im} dV^n$, so $W(R)$ is also a strict Dieudonné complex. (Next week, Rahul will tell us about how the saturation and completion operations behave for Dieudonné algebras.)

3.1 The de Rham complex

As the following proposition shows, a more interesting example of a Dieudonné algebra is given by the absolute de Rham complex $\Omega_R^* := \Omega_{R/\mathbb{Z}}^*$ of a (p-torsion-free) ring R equipped with a lift of Frobenius. (Remark: you can drop the p-torsion-free hypothesis by using δ-maps instead of lifts of Frobenius, and taking some care with the setup. This is done in section 3.7 of the paper, but we will ignore it.)

Proposition 3.3. *Let R be a p-torsion-free commutative ring, and $\varphi : R \to R$ a homomorphism with $\varphi(x) \equiv x^p \pmod{p}$. Then there is a unique ring homomorphism $F : \Omega_R^* \to \Omega_R^*$ such that*

1. *For each $x \in R = \Omega_R^0$, we have $F(x) = \varphi(x)$.*
2. *For each $x \in R$, we have $F(dx) = x^{p-1}dx + d(\frac{\varphi(x)-x^p}{p})$.*

This F gives Ω_R^* the structure of a Dieudonné algebra.

Interpretation of the second condition: F is the "divided Frobenius" $\frac{\varphi^*}{p^n}$ on Ω_R^n, where $\varphi^* = \alpha_F$ is pullback of differential forms. As an aside, note that we could alternatively define the cdga homomorphism $\alpha : \Omega_R^* \to \Omega_R^*$ to be pullback of differential forms, note by the calculation

$$\alpha(dx) = d\alpha(x) = d\varphi(x) \equiv d(x^p) = px^{p-1} \equiv 0 \pmod{p} \tag{15}$$

that α is divisible by p on Ω_R^1, and conclude (because Ω^* is generated by Ω^1 as a ring) that α is divisible by p^n on Ω^n. Then if Ω_R^* is p-torsion-free, we could define F as $\frac{\alpha}{p^n}$ on Ω^n. But in fact Ω_R^* can have p-torsion even if R is quite reasonable. For example, if K is a number field, then $\Omega^1_{\mathcal{O}_K/\mathbb{Z}}$ has p-torsion if and only if $K/\mathbb{Q}$ is ramified at p. (Its annihilator is the different ideal of $K/\mathbb{Q}$.)

The proof of the proposition is explicit. Uniqueness of F is clear, as it has been specified on a set of generators of Ω_R^*. One calculates that the map $\rho : R \to \Omega_R^1$ given by $x^{p-1}dx + d(\frac{\varphi(x)-x^p}{p})$ is a φ-linear derivation; i.e. an additive map satisfying $\rho(xy) = \varphi(x)\rho(y) + \varphi(y)\rho(x)$. The universal property of $d : R \to \Omega_R^1$ implies that ρ factors as $F \circ d$ for some φ-semilinear map $F : \Omega_R^1 \to \Omega_R^1$. Then F satisfies the relations $(F(dx))^2 = 0$ for all $x \in R$, so it extends to the exterior algebra Ω_R^*. Finally, one checks that it satisfies the necessary relation $dF = pFd$ on the inputs x and dx ($x \in R$), and that this identity is preserved by multiplication.

Proposition 3.4. *Let R and φ be as before, and view Ω_R^* as a Dieudonné algebra. For any p-torsion-free Dieudonné algebra A^*, the restriction map*

$$\mathrm{Hom}_{\mathbf{DA}}(\Omega_R^*, A^*) \to \mathrm{Hom}_{\mathrm{ring}}(R, A^0) \tag{16}$$

is injective, and its image consists of homomorphisms $f : R \to A^0$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & A^0 \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle F} \\ R & \xrightarrow{f} & A^0 \end{array}$$

commutes.